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On a sequence related to that of Thue–Morse and its applications

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Abstract

It is known that the sequence $1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 2, \ldots$ of lengths of blocks of identical symbols in the Thue–Morse sequence has several extremal properties among all non-periodic sequences of the symbols 1 and 2. Its generating function $W(x)$ is equal to $\prod_{k=1}^{\infty}(1+x^{(2^k+(-1)^{k-1})/3})$. In terms of combinatorics on words, for any given $x \in (0, 1)$ and $\varepsilon > 0$, we prove that every non-periodic word of an alphabet $\{1, 2\}$ has a suffix $\mathbf{s}$ whose generating function $S(x)$ satisfies the inequality $xS(-x) > 1 - W(-x) - \varepsilon$. Using this, we prove several bounds for the largest and the smallest limit points of the sequence of fractional parts $\{\xi b^n\}$, $n = 0, 1, 2, \ldots$, where $b < -1$ is a negative rational number and ξ is a real number. Our results show, for example, that, for any real number $\xi \neq 0$, the sequence of fractional parts $\{\xi(-3/2)^n\}$, $n = 0, 1, 2, \ldots$, has a limit point greater than 0.466452. Furthermore, for each integer $b \leqslant -2$ and each real number $\xi \notin \mathbb{Q}$, we prove that $\liminf_{n\to\infty}\{\xi b^n\} \leqslant \prod_{k=1}^{\infty}(1-|b|^{-(2^k+(-1)^{k-1})/3})$ and show that this inequality is sharp.

MSC: 11A63; 11K06; 68R15

Keywords: Thue–Morse sequence; Distribution modulo 1; Radix representation; Words

1. Introduction

In this paper, we consider the sequence

$$\mathbf{w} : w_0, w_1, w_2, w_3, \ldots = 1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 2, \ldots \tag{1}$$

of the number of consecutive identical symbols in the Thue–Morse sequence

$$\mathbf{t} : t_0, t_1, t_2, t_3, \ldots = 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots \tag{2}$$

which is defined by $t_0 = 0$, $t_{2k+1} = 1 - t_k$ and $t_{2k} = t_k$. In Sloane's online encyclopedia of integer sequences http://www.research.att.com/∼njas/sequances/ the numbers $A026465$ and $A001285$ are assigned to the sequences (1) and (2), respectively. Both sequences, especially (2), have a long history of research. See, for instance, the review paper [10], where several applications of various properties of (2) are described and an impressive list of references is given. The sequence of partial sums of (1), namely, $1, 3, 4, 5, 7, 9, 11, 12, 13, \ldots$ is also well known. It is the lexicographically smallest set of integers A which contains 1 and no elements a and b such that $b = 2a$, so that A and $2A$ form a partition of the set of positive integers $\mathbb{N}$. Its properties have been investigated in [7] (see also [10] for more references).

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We shall prove below (see Lemma 5) that, for any $x \in \mathbb{C}$ satisfying $|x| < 1$, the generating function of (1) is given by the formula

$$W(x) = \sum_{j=0}^{\infty} w_j x^j = \prod_{k=1}^{\infty} (1 + x^{(2^k+(-1)^{k-1})/3}). \tag{3}$$

Note that the exponents $f_k = (2^k + (-1)^{k-1})/3 \in \mathbb{N}$ in (3) satisfy the linear recurrence relation

$$f_1 = f_2 = 1, \quad f_k = f_{k-1} + 2f_{k-2} \quad \text{for } k \geqslant 3. \tag{4}$$

It should be said that (3) is equivalent to the main result of [7], where the authors used the formula $f_{k+1} = 2f_k + (-1)^k$ instead of (4). (It is easily seen that this leads to the same formula $f_k = (2^k + (-1)^{k-1})/3$, although it was not given explicitly in [7].)

Set

$$F(x) = W(-x) = (1-x)^2(1-x^3)(1-x^5)(1-x^{11})\cdots = \prod_{k=1}^{\infty} (1 - x^{(2^k+(-1)^{k-1})/3}). \tag{5}$$

In this paper, we shall prove the following:

Theorem 1. *Let $K \geqslant 2$ be a fixed integer. Suppose that the sequence $s_0, s_1, s_2, \ldots \in \{1, 2, \ldots, K\}$ is non-periodic. If $x \in (0, 1)$ then, for any $\varepsilon > 0$, there are infinitely many $n \in \mathbb{N}$ such that*

$$\sum_{j=0}^{\infty} s_{n+j}(-x)^j > (1 - F(x))/x - \varepsilon. \tag{6}$$

Recall that the sequence $s_0, s_1, s_2, \ldots$ is called *periodic* (or *ultimately periodic*) if there is a $t \in \mathbb{N}$ such that $s_n = s_{n+t}$ for each $n \geqslant n_0$.

In Section 3, we will prove Theorem 1 for $x \in (0, 1/2]$ and show that the inequality (6) is sharp in the sense that one cannot remove ε from the right-hand side of (6). In order to show this we shall take $s_k = w_{k+2}$ for each $k \geqslant 0$. However, the proof of the theorem is quite involved when x is close to 1.

This paper is organized as follows. In the next section, we give some number theoretic applications of Theorem 1. In Section 3, we shall restate Theorem 1 for words (instead of sequences) and remind some basic results concerning the sequence (1). In Section 4, we obtain certain inequalities for polynomials related to the functions $W(x)$ and $F(x)$. The proof of Theorem 1 will be given in Section 5. In Section 6, we give two auxiliary results, both of independent interest. Combined with Theorem 1 they easily imply all our number theoretic results (see Section 7). Lemma 6 and its more general version (14) seem to be of independent interest too, because the proofs of purely analytical inequalities are based on the properties of the Thue–Morse sequence.

2. Fractional parts of powers of a negative rational number

Let us begin with the main application of Theorem 1, where, by (5), we have $F(x) = \prod_{k=1}^{\infty} (1 - x^{(2^k+(-1)^{k-1})/3})$.

Theorem 2. *Let $b = -p/q$, where $p > q > 1$ are two coprime positive integers. Then, for any real number ξ, the sequence of fractional parts $\{\xi b^n\}$, $n = 0, 1, 2, \ldots$, has a limit point $\leqslant 1 - (1 - F(q/p))/q$; also, if $\xi \neq 0$ then it has a limit point $\geqslant (1 - F(q/p))/q$.*

The distribution of the sequence of fractional parts $\{\xi\alpha^n\}$, $n = 0, 1, 2, \ldots$, where α is a real number of modulus greater than 1 and ξ is a non-zero real number, in the interval [0, 1) is a well-known unsolved problem. For any fixed α and 'random' ξ and fixed $\xi \neq 0$ and 'random' α the sequence $\{\xi\alpha^n\}$, $n = 0, 1, 2, \ldots$, is uniformly distributed in [0, 1), by the results of Weyl [40] and Koksma [31], respectively. On the other hand, for any fixed α, where $\alpha > 1$, there exist 'exceptional' ξ such that the fractional parts $\{\xi\alpha^n\}$, $n = 0, 1, 2, \ldots$, are not even everywhere dense in [0, 1). See [18,34] for such results. They answer a corresponding question of Erdös [25] and have applications in graph theory [4,30,35].

There are, of course, infinitely many pairs of real numbers ξ, α for which one can consider the behavior of the sequence $\{\xi\alpha^n\}$, $n = 0, 1, 2, \ldots$. However, the best known pair in this context is $\xi = 1$, $\alpha = 3/2$. The upper bound $\{(3/2)^n\} < 1 - (3/4)^n$ (if proved for each integer $n \geqslant 5$) would complete one of the versions of Waring's problem [38]. On the other hand, Vijayaraghavan's question [39] on whether the sequence $\{(3/2)^n\}$, $n = 0, 1, 2, \ldots$, has infinitely many elements in both $(0, 1/2)$ and $(1/2, 1)$ is also open. Mahler's question [33] on whether there is a $\xi > 0$ such that $\{\xi(3/2)^n\} < 1/2$ for each $n \in \mathbb{N}$ is still open too. Both Mahler's and Vijayaraghavan's conjectures would follow if, for any $\xi \neq 0$, one could prove that the interval $(1/2, 1]$ contains a limit point of the sequence $\{\xi(3/2)^n\}$, $n = 0, 1, 2, \ldots$. The strongest result in this direction is that, for any $\xi \neq 0$, the interval $[1/3, 1]$ must contain at least one limit point of the sequence $\{\xi(3/2)^n\}$, $n = 0, 1, 2, \ldots$ (see [27]).

It is therefore of interest to give the following numerical version of Theorem 2 for $b = -3/2$. Since $F(2/3) = \prod_{k=1}^{\infty}(1 - (2/3)^{(2^k+(-1)^{k-1})/3}) = 0.0670944\ldots$ is quite small, we see that in this case one comes closer to the required interval $(1/2, 1]$.

Corollary 3. *For any real number $\xi \neq 0$, the sequence of fractional parts $\{\xi(-3/2)^n\}$, $n = 0, 1, 2, \ldots$, has a limit point smaller than* 0.533547 *and a limit point greater than* 0.466452.

In Theorem 2 we did not consider the case $q = 1$, that is, when $b \leqslant -2$ is an integer. For $\xi \in \mathbb{Q}$ and $b \in \mathbb{Z}$, it is easily seen that $\{\xi b^n\}$, $n = 0, 1, 2, \ldots$, is a periodic sequence taking only finitely many values. Therefore, it is natural to add the extra condition $\xi \notin \mathbb{Q}$ in the case when b is an integer. We then obtain the following result.

Theorem 4. *Let $b \leqslant -2$ be an integer. Then, for any real irrational number ξ, we have*

$$\liminf_{n\to\infty} \{\xi b^n\} \leqslant F(-1/b) \quad \text{and} \quad \limsup_{n\to\infty} \{\xi b^n\} \geqslant 1 - F(-1/b), \tag{7}$$

where $F(-1/b) = \prod_{k=1}^{\infty}(1-(1/|b|)^{(2^k+(-1)^{k-1})/3})$. These bounds are sharp: for example, by setting $\xi_b = -bF(-1/b)$, we have that (i) *the number ξ_b is transcendental, and* (ii) *the inequalities $\{-\xi_b b^n\} > F(-1/b)$ and $\{\xi_b b^n\} < 1 - F(-1/b)$ hold for each integer $n \geqslant 0$, hence the inequalities in* (7) *cannot be replaced by strict inequalities.*

The distribution of fractional parts $\{\xi b^n\}$, $n = 0, 1, 2, \ldots$, where $b \geqslant 2$ is a positive integer and $\xi \notin \mathbb{Q}$, have been considered in [14]. The case when $\xi \neq 0$ and $b > 1$ is a rational non-integer number was considered in [3,5,13,23,26,27,36]. In [20], $b > 1$ is allowed to be algebraic, whereas the interval constructions of [4,37] allow b to be transcendental. In fact, by the same method as in [37], one can prove that, for any (positive or negative) real number b satisfying $|b| > 2$, there is a non-zero real number ξ such that $\{\xi b^n\} < 1/(|b| - 1)$ for each integer $n \geqslant 0$. This shows that the constants of Theorem 2 cannot be improved too much if q is 'small' and p is 'large'. For $q = 2$, by the inequality $(1 - F(2/p))/2 > 2/p - 2/p^2$, Theorem 2 implies that the sequence $\{\xi(-p/2)^n\}$, $n = 0, 1, 2, \ldots$, where $p \geqslant 3$ is odd and $\xi \neq 0$, has a limit point greater than $2/p - 2/p^2$. On the other hand, there is a real number $\xi \neq 0$ such that $\{\xi(-p/2)^n\} < 2/(p - 2)$ for each $n \geqslant 0$.

All results stated in this section remind the corresponding results for the distance to the nearest integer $\|\xi(p/q)^n\|$ from our paper [21], where the product $\prod_{k=1}^{\infty}(1 - x^{2^k})$ was playing a role similar to that of $F(x) = \prod_{k=1}^{\infty}(1 - x^{(2^k+(-1)^{k-1})/3})$ here. Note that $\|\xi(-p/q)^n\| = \|\xi(p/q)^n\|$, so the results of [21] for the distances to the nearest integer hold for the powers of a negative rational number too.

For $b = -2$, we calculate $F(1/2) = \prod_{k=1}^{\infty}(1 - 2^{-(2^k+(-1)^{k-1})/3}) = 0.2118104\ldots$, hence

$$\liminf_{n\to\infty} \{\xi(-2)^n\} < 0.211811 \quad \text{and} \quad \limsup_{n\to\infty} \{\xi(-2)^n\} > 0.788189$$

for any $\xi \notin \mathbb{Q}$. In particular, the upper bound for the smallest limit point of $\{\xi(-2)^n\}$, $n = 0, 1, 2, \ldots$, (which is $< 1/4$) implies that, for any $\xi \notin \mathbb{Q}$, we have $[3\xi(-2)^n] = 3[\xi(-2)^n]$ and $[4\xi(-2)^n] = 4[\xi(-2)^n]$ for infinitely many $n \in \mathbb{N}$. Thus, given any $\xi \notin \mathbb{Q}$, the sequence of integral parts $[\xi(-2)^n]$, $n = 0, 1, 2, \ldots$, contains infinitely many numbers divisible by 3 and infinitely many numbers divisible by 4. (Evidently, for $\xi = 1$, we obtain that the numbers $[(-2)^n] = (-2)^n$, where $n \in \mathbb{N}$, are not divisible by 3.) Likewise, since $F(1/3) = \prod_{k=1}^{\infty}(1 - 3^{-(2^k+(-1)^{k-1})/3}) = 0.426219\ldots$, we get that $\liminf_{n\to\infty}\{\xi(-3)^n\} < 0.42622$ for $\xi \notin \mathbb{Q}$. So, for any real number $\xi \notin \mathbb{Q}$, the sequence of integral parts $[\xi(-3)^n]$, $n = 0, 1, 2, \ldots$, contains infinitely many even numbers.

The only result of this kind which can be obtained for the powers of positive integers is that, for any real ξ, the integral parts $[\xi 2^n]$ are even for infinitely many $n \in \mathbb{N}$. See [24] and also [15,19,28], Problem E19 in [29,42] for other results concerning prime and composite numbers in the sequence of integral parts $[\xi\alpha^n]$, $n = 0, 1, 2, \dots$.

3. Exploring the minimality of the word a = 2112221121121122 . . .

Below, we shall frequently use the terminology of combinatorics on words. See, for instance, [11] or [12] for an introduction to this subject. Throughout, any finite set is referred to as an *alphabet*. Sequences (finite or infinite) of letters of an alphabet are called *words*. For any (finite or infinite) word **s**, its beginning is called a *prefix*, its end is called a *suffix*, and each block of consecutive symbols of **s** is called a *factor*. For example, 21 occurs in the word 2121221 of the alphabet $\{1, 2\}$ as its prefix, as its factor and as its suffix too. In addition, we call the factor **u** of **s** a *proper factor* if there exist two non-empty words $\mathbf{u}'$ and $\mathbf{u}''$ such that $\mathbf{s} = \mathbf{u}'\mathbf{u}\mathbf{u}''$. For instance, 21 is a proper factor of 2121221, but is not a proper factor of 21221. Clearly, each proper factor of an infinite word is a finite word. Finally, if $\mathbf{s} = s_0s_1s_2\dots$, then each suffix $s_ks_{k+1}s_{k+2}\dots$, where $k > 0$, is called a *proper suffix* of **s**.

If $\mathbf{s} = s_0s_1s_2s_3\dots$ is a word of an alphabet which is a finite subset of the set of real numbers $\mathbb{R}$ then, for any given $x \in \mathbb{C}$, where $|x| < 1$, we put

$$\mathbf{s}(x) = s_0 - s_1x + s_2x^2 - s_3x^3 + \cdots,$$

where the sum is finite or infinite depending on whether the word **s** is finite or infinite. For example, if $\mathbf{s} = 21122$ then $\mathbf{s}(x) = 2 - x + x^2 - 2x^3 + 2x^4$. Of course, $\mathbf{s}(x) = S(-x)$, where $S(x) = s_0 + s_1x + s_2x^2 + \cdots$ is the generating function of the sequence $s_0, s_1, s_2, \dots$. Finally, if **s** is a finite word, then by $l(\mathbf{s})$ we shall denote the length of **s**, so that $\deg \mathbf{s}(x) = l(\mathbf{s}) - 1$.

With this terminology, Theorem 1 is equivalent to the following statement: if $x \in (0, 1)$ and $\varepsilon > 0$ are fixed, then any infinite non-periodic word of an alphabet $\{1, 2, \dots, K\}$, where $K \geqslant 2$, has a proper suffix **s** such that

$$x\mathbf{s}(x) > 1 - F(x) - \varepsilon. \tag{8}$$

We shall show first that one can restrict the alphabet $\{1, 2, \dots, K\}$ to the alphabet $\{1, 2\}$. Indeed, by removing the first symbol of the initial word we shall obtain its proper suffix. If the symbol k, where $k \geqslant 3$, occurs in this suffix, then it has suffixes **s** and $\mathbf{s}' = k\mathbf{s}$. Both **s** and $\mathbf{s}'$ are proper suffixes of the initial word. Hence $\mathbf{s}'(x) = k - x\mathbf{s}(x)$ which implies that

$$\max\{\mathbf{s}(x), \mathbf{s}'(x)\} \geqslant k/(1 + x) \geqslant 3/(1 + x).$$

In order to prove (8) we need to show that $3/(1 + x) > (1 - F(x))/x$, that is,

$$2x + (1 + x)F(x) > 1. \tag{9}$$

Indeed, by (5), we see that $F(x) > 0$ for each $x \in (0, 1)$, so (9) obviously holds for $x \in [1/2, 1)$. On the other hand, for $x \in (0, 1/2)$, by (1), (3) and (5), we obtain that

$$2x + (1 + x)F(x) - 1 > 2x + F(x) - 1 = w_2x^2 - w_3x^3 + w_4x^4 - w_5x^5 + \cdots > 0,$$

because $w_{2j}x^{2j} - w_{2j+1}x^{2j+1} \geqslant x^{2j}(1 - 2x) > 0$. This implies (9).

This argument works if any of the symbols $k \in \{3, \dots, K\}$ occurs in the initial word infinitely often. So it suffices to prove (8) for non-periodic words of an alphabet $\{1, 2\}$. As we already said above, the proof is quite easy in the case when $x \in (0, 1/2]$. We will give the proof of this statement at the end of this section. (This is sufficient for the proof of Theorem 4, but is not sufficient for the proof of Theorem 2 when $p < 2q$.)

There are several ways to produce the word **w** defined in (1). For example, one can start with 1 and then at each step replace each 1 by 121 and each 2 by 12221. Since for us it is more convenient to work the word $\mathbf{a} = w_1w_2w_3\dots$ (without the first symbol $w_0 = 1$), let us put

$$\mathbf{w} = 1\mathbf{a}. \tag{10}$$

There are several definitions of **a**. As above, it is the fixed point of $2 \to 211$, $1 \to 2$. On the other hand, set $\mathbf{a}_0 = 1$, $\mathbf{a}_1 = 2$ and $\mathbf{a}_k = \mathbf{a}_{k-1}\mathbf{a}_{k-2}^2 = \mathbf{a}_{k-1}\mathbf{a}_{k-2}\mathbf{a}_{k-2}$ for each $k \geqslant 2$. Then the word **a** is obtained as a limit of $\mathbf{a}_k$ as $k \to \infty$, that is, $\mathbf{a} = \lim_{k\to\infty} \mathbf{a}_k$ (see [21]). This definition gives the recurrent formula $l(\mathbf{a}_k) = l(\mathbf{a}_{k-1}) + 2l(\mathbf{a}_{k-2})$, where $l(\mathbf{a}_0) = l(\mathbf{a}_1) = 1$. So, by (4), we obtain that $l(\mathbf{a}_k) = f_{k+1}$, where $f_k = (2^k + (-1)^{k-1})/3$.

For any two distinct words $\mathbf{u} = u_0u_1u_2\ldots$ and $\mathbf{v} = v_0v_1v_2\ldots$ of the alphabet $\{1, 2\}$ such that neither is a prefix of another, let $k \geqslant 0$ be the smallest integer for which the numbers u_k and v_k are distinct, say, $u_k=2$, $v_k=1$. We then introduce the order $>$ on the set of such words, where $\mathbf{u} > \mathbf{v}$ if k is even and $\mathbf{v} > \mathbf{u}$ if k is odd. For instance, $2121 > 2122122\ldots$, because the first pair of distinct symbols 1 and 2 occurs at fourth place, i.e. $k=3$. For each $x \in (0, 1/2]$, this definition of order implies that $\mathbf{u}(x) \geqslant \mathbf{v}(x)$ if $\mathbf{u} > \mathbf{v}$. See, for instance, [16] for the extremal properties of this order.

Fix a positive integer m which is so large that $\mathbf{a}_m(x) > \mathbf{a}(x) + 2x^{f_{m+1}} - \varepsilon$. Corollary 4 of [21] asserts that if m is a positive integer, then any non-periodic word of the alphabet $\{1, 2\}$ either contains the factor $\mathbf{a}_m$ or it contains a finite factor $\mathbf{u}$ that satisfies $\mathbf{u} > \mathbf{a}$. Furthermore, by adding an extra symbol 1 or 2 to the right of $\mathbf{u}$, we have that $\mathbf{u}1 > \mathbf{a}$ and $\mathbf{u}2 > \mathbf{a}$. So we can assume without loss of generality that $l(\mathbf{u})$ is even.

We remark that the fact that each non-periodic word contains either $\mathbf{a}_m$ or a finite factor $\mathbf{u}$ satisfying $\mathbf{u} > \mathbf{a}$ follows from the following result: the word $\mathbf{a}$ is the smallest non-periodic infinite word of the alphabet $\{1, 2\}$ which is greater than its arbitrary proper suffix with respect to the order $>$. Although this was proved already in [6,8], see also [9,21,32] for other proofs.

Let $\mathbf{s}$ be any infinite word which is a proper suffix of the initial non-periodic word such that its prefix is $\mathbf{a}_m$ (the first case) or the word $\mathbf{u}$ of even length satisfying $\mathbf{u} > \mathbf{a}$ (the second case). In the first case, using $\mathbf{a}_m(x) > \mathbf{a}(x) + 2x^{f_{m+1}} - \varepsilon$ and $l(\mathbf{a}_m) = f_{m+1} = \deg \mathbf{a}_m(x) + 1$, we see that, for $x \in (0, 1/2]$,

$$\mathbf{s}(x) \geqslant \mathbf{a}_m(x) - 2x^{f_{m+1}} > \mathbf{a}(x) - \varepsilon.$$

In the second case, $\mathbf{s}(x) \geqslant \mathbf{u}(x) \geqslant \mathbf{a}(x)$. Summarizing both cases, we conclude that any non-periodic word of $\{1, 2\}$ contains a suffix $\mathbf{s}$ such that $\mathbf{s}(x) > \mathbf{a}(x) - \varepsilon$. Since, by (5) and (10), we have $\mathbf{a}(x) = (1 - F(x))/x$, this completes the proof of (8) for all $x \in (0, 1/2]$. The proof of (8) for $x \in (1/2, 1)$ is much more subtle, because it can happen that $\mathbf{u}(x) < \mathbf{v}(x)$ although $\mathbf{u} > \mathbf{v}$.

On account of the above result on the minimality of $\mathbf{a} = w_1w_2w_3\ldots$, the word $w_2w_3w_4\ldots$ itself and its all suffixes are smaller than $\mathbf{a}$. So, for each suffix $\mathbf{s}$ of $w_2w_3w_4\ldots$ and each real number $x \in (0, 1/2]$, we have that $\mathbf{s}(x) < \mathbf{a}(x) = (1 - F(x))/x$. This shows that one cannot omit ε in the right-hand side of (6) or (8).

4. Inequalities for the factors of the word a

Recall that $l(\mathbf{a}_k) = f_{k+1}$.

Lemma 5. *We have* $1\mathbf{a}_k(x) = 1 - x\mathbf{a}_k(x) = (1 - x^{f_1})\ldots(1 - x^{f_k}) - x^{f_{k+1}}$. *In particular, for each* $|x| < 1$, *letting* $k \to \infty$, *by* (10), *this yields* $\mathbf{w}(x) = \prod_{k=1}^{\infty}(1 - x^{f_k})$ *which, by* (5), *implies* (3).

Proof. This is clear for $k = 1$ and $k = 2$. From $\mathbf{a}_k = \mathbf{a}_{k-1}\mathbf{a}_{k-2}^2$ it follows that

$$\mathbf{a}_k(x) = \mathbf{a}_{k-1}(x) - x^{f_k}\mathbf{a}_{k-2}(x) + x^{f_k+f_{k-1}}\mathbf{a}_{k-2}(x) = \mathbf{a}_{k-1}(x) - x^{f_k}(1 - x^{f_{k-1}})\mathbf{a}_{k-2}(x).$$

Multiplying by x and using the inductive hypothesis for $j < k$ which, in particular, implies that $x\mathbf{a}_j(x) = -(1 - x^{f_1})\ldots(1 - x^{f_j}) + 1 + x^{f_{j+1}}$ for $j = k - 1$ and $j = k - 2$, we get

$$\begin{aligned} x\mathbf{a}_k(x) &= 1 + x^{f_k} - (1 - x^{f_1})\ldots(1 - x^{f_{k-1}}) - x^{f_k}(1 - x^{f_{k-1}})(1 + x^{f_{k-1}} - (1 - x^{f_1})\ldots(1 - x^{f_{k-2}})) \\ &= 1 + x^{f_k+2f_{k-1}} - (1 - x^{f_1})\ldots(1 - x^{f_k}) = 1 + x^{f_{k+1}} - (1 - x^{f_1})\ldots(1 - x^{f_k}). \end{aligned}$$

Thus $1\mathbf{a}_k(x) = 1 - x\mathbf{a}_k(x) = (1 - x^{f_1})\ldots(1 - x^{f_k}) - x^{f_{k+1}}$, as claimed. The second statement for $|x| < 1$ follows immediately, because $\mathbf{w} = 1\mathbf{a}$ and $\mathbf{a}(x) = \lim_{k\to\infty}\mathbf{a}_k(x)$. $\square$

Lemma 6. *Let* $b_1 < b_2 < b_3 < \cdots$ *be a sequence of positive integers satisfying* $b_1 + b_2 + \cdots + b_{k-1} < b_k$ *for each* $k \geqslant 2$. *Then*

$$\prod_{k=1}^{\infty}(1 - x^{b_k}) > 1 - x^{b_1} - x^{b_2} + x^{b_1+b_2} - x^{b_3}$$

for $x \in (0, 1)$.

Proof. Fix $x \in (0, 1)$. By expanding the product $\prod_{k=1}^{\infty}(1 - x^{b_k})$ into an infinite sum, we derive that

$$\prod_{k=1}^{\infty}(1 - x^{b_k}) = 1 - x^{b_1} - x^{b_2} + x^{b_1+b_2} - x^{b_3} + x^{b_1+b_3} + x^{b_2+b_3} - x^{b_1+b_2+b_3} - x^{b_4} + x^{b_1+b_4} + \cdots . \tag{11}$$

Observe that the sequence of signs $+$ and $-$, that is, $+--+-++--+\cdots$ in (11) is independent of the choice of the sequence $b_1 < b_2 < b_3 < \cdots$, because $b_1 + \cdots + b_{k-1} < b_k$. In particular, choosing $b_k = 2^{k-1}$, $k = 1, 2, \ldots$, we will get the same sequence of $+$ and $-$ as in the case of an arbitrary sequence $b_1 < b_2 < b_3 < \cdots$ satisfying the condition of the lemma. For this particular sequence of the powers of 2, it is known (see, e.g., [10]) that $\prod_{k=1}^{\infty}(1-x^{2^{k-1}}) = \sum_{j=0}^{\infty}(-1)^{t_j}x^j$, where $t_0, t_1, t_2 \ldots$ is the Thue–Morse sequence defined in (2).

Since $1 - x^{b_1} - x^{b_2} + x^{b_1+b_2} - x^{b_3}$ represent the first five summands in (11) corresponding to the prefix $+--+-$, it suffices to show that the suffix of this word $(-1)^{t_5}(-1)^{t_6}(-1)^{t_7}\ldots$, namely,

$$++--++-+--+-++-+--++--+-++-\cdots \tag{12}$$

can be divided into a sequence of just three blocks $+-$, $++--$ and $++-+--$. Combined with (11) this would clearly imply that the difference between the product $\prod_{k=1}^{\infty}(1 - x^{b_k})$ and the polynomial $1 - x^{b_1} - x^{b_2} + x^{b_1+b_2} - x^{b_3}$ corresponding to the block $+--+-$ is positive, because the polynomials corresponding to the blocks $+-$, $++--$, $++-+--$ are positive for each $x \in (0, 1)$.

Note that the sequence (12) starts with $++--$. Assume that at certain place (say, at ℓth, that is, after using $\ell - 1$ elements of (12) which are divided into a sequence of three blocks mentioned above) we obtain a block which is neither $+-$, nor $++--$, nor $++-+--$. Put

$$\Lambda(k) = (-1)^{t_0} + \cdots + (-1)^{t_k}. \tag{13}$$

Since $\Lambda(k)$ takes the values $-1, 0, 1$ only and since $\Lambda(4) = \Lambda(3 + \ell) = -1$, the ℓth element of (12) is $+$. In case the next after $+$ is $-$, we can stop: the block is $+-$. Suppose the block which starts at ℓth place is $++$.

Recall that the Thue–Morse sequence is cube free and overlap free, i.e., contains no block of the form *awawa*, where $a \in \{+, -\}$ and w is a finite block of $+$ and $-$. In particular, this implies that the next element after $++$ must be $-$. In case the next is $-$ we obtain the block $++--$, a contradiction. So suppose the block is $++-+$. From $\Lambda(7 + \ell) = 1$, we see that the next element in (12) is $-$. We thus have the block $++-+-$. It follows that the next element must be $-$, because the block $+-+-+$ cannot occur as a factor of the Thue–Morse sequence which is overlap free! So the block starting at the ℓth place of (12) is $++-+--$, a contradiction. This completes the proof of the lemma. □

Let us write

$$\prod_{k=1}^{\infty}(1 - x^{b_k}) = \sum_{j=0}^{\infty}(-1)^{t_j}x^{B_j},$$

where $b_1 < b_2 < b_3 < \cdots$ is a sequence of integers satisfying the condition $b_1 + \cdots + b_{k-1} < b_k$, the sequence $t_0, t_1, t_2, \ldots$ is (2) and $B_0 = 0 < B_1 = b_1 < B_2 = b_2 < B_3 = b_1 + b_2 < \cdots$. Then, by the same argument as in the proof of Lemma 6, we obtain the following, more general, inequality

$$\prod_{k=1}^{\infty}(1 - x^{b_k}) > \sum_{j=0}^{m}(-1)^{t_j}x^{B_j}, \tag{14}$$

where m is an arbitrary positive integer for which $\Lambda(m) = -1$ (see (13)). For example, $\Lambda(8) = -1$, so, setting $b_k = 3^{k-1}$, $k = 1, 2, \ldots$, by (14), we derive that

$$(1 - x)(1 - x^3)(1 - x^9)(1 - x^{27})(1 - x^{81})\cdots > 1 - x - x^3 + x^4 - x^9 + x^{10} + x^{12} - x^{13} - x^{27}$$

for any $x \in (0, 1)$.

In the proof of the next lemma we shall use the equality $l(\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k) = 2f_{k+1} + f_k$.

Lemma 7. *If k is a positive integer then $\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k(x) > (1 + x^{2f_{k+1}+f_k})\mathbf{a}(x)$ for $x \in (0, 1)$.*

Proof. By Lemma 5, $\mathbf{a}_k(x) = (1 + x^{f_{k+1}} - P_k(x))/x$, where $P_k(x) = (1 - x^{f_1}) \dots (1 - x^{f_k})$. Using the fact that the numbers f_k, where $k \geqslant 1$, are all odd, we obtain that $x\mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k(x)$ is equal to

$$\begin{aligned}&1 + x^{f_{k+1}} - P_k(x) - x^{f_{k+1}}(1 + x^{f_k} - P_{k-1}(x)) + x^{f_{k+1}+f_k}(1 + x^{f_{k+1}} - P_k(x))\\ &\quad = 1 + x^{2f_{k+1}+f_k} + x^{f_{k+1}}P_{k-1}(x) - (1 + x^{f_{k+1}+f_k})P_k(x).\end{aligned}$$

Similarly, using $F(x) = 1 - x\mathbf{a}(x) = \prod_{k=1}^{\infty}(1 - x^{f_k})$, we obtain that

$$x(1 + x^{2f_{k+1}+f_k})\mathbf{a}(x) = (1 + x^{2f_{k+1}+f_k})\left(1 - \prod_{k=1}^{\infty}(1 - x^{f_k})\right).$$

Subtracting $1 + x^{2f_{k+1}+f_k}$ from both sides, we see that the inequality of the lemma is equivalent to the inequality

$$x^{f_{k+1}}P_{k-1}(x) - (1 + x^{f_{k+1}+f_k})P_k(x) + (1 + x^{2f_{k+1}+f_k})\prod_{k=1}^{\infty}(1 - x^{f_k}) > 0.$$

Observing that $P_k(x) = (1 - x^{f_k})P_{k-1}(x)$ and using (4) we transform the above inequality into

$$(1 + x^{2f_{k+1}+f_k})\prod_{k=1}^{\infty}(1 - x^{f_k}) > (1 - x^{f_k} - x^{f_{k+1}} + x^{f_{k+1}+f_k} - x^{f_{k+2}})P_{k-1}(x).$$

Next, dividing both sides by $P_{k-1}(x)$, we obtain the following equivalent inequality

$$(1 + x^{2f_{k+1}+f_k})\prod_{j=k}^{\infty}(1 - x^{f_j}) > 1 - x^{f_k} - x^{f_{k+1}} + x^{f_{k+1}+f_k} - x^{f_{k+2}}.$$

So it suffices to prove the next, stronger, inequality

$$\prod_{j=k}^{\infty}(1 - x^{f_j}) > 1 - x^{f_k} - x^{f_{k+1}} + x^{f_{k+1}+f_k} - x^{f_{k+2}}. \tag{15}$$

Indeed, for $k \geqslant 2$, the inequality (15) follows directly from Lemma 6, because $f_k + \dots + f_{k+j} < f_{k+j+1}$ for any pair of integers $k \geqslant 2$, $j \geqslant 0$. This follows, for example, by observing that $f_1 + f_2 + \dots + f_k$ is equal to f_{k+1} if k is odd and to $f_{k+1} - 1$ if k is even.

For $k = 1$, we shall apply Lemma 6 to the product $\prod_{k=2}^{\infty}(1 - x^{f_k})$ first. It follows that this product is greater than $1 - x^{f_2} - x^{f_3} + x^{f_2+f_3} - x^{f_4} > 1 - x - x^3$. Hence

$$\prod_{k=1}^{\infty}(1 - x^{f_k}) > (1 - x)(1 - x - x^3) = 1 - 2x + x^2 - x^3 + x^4 > 1 - x^{f_1} - x^{f_2} + x^{f_1+f_2} - x^{f_3},$$

which implies (15) for $k = 1$. $\square$

Next, we shall use the equality $l(\mathbf{a}_k\mathbf{a}_{k-1}^4) = f_{k+1} + 4f_k$.

Lemma 8. *If k is a positive integer then $\mathbf{a}_k\mathbf{a}_{k-1}^4(x) > (1 + x^{f_{k+1}+4f_k})\mathbf{a}(x)$ for $x \in (0, 1)$.*

Proof. Once again we shall use the equalities $\mathbf{a}_k(x) = (1 + x^{f_{k+1}} - P_k(x))/x$, where $P_k(x) = (1 - x^{f_1}) \dots (1 - x^{f_k})$ for $k \in \mathbb{N}$, and the fact that the numbers f_k, $k = 1, 2, \dots$, are all odd. It follows that $x\mathbf{a}_k\mathbf{a}_{k-1}^4(x)$ is equal to

$$\begin{aligned}&1 + x^{f_{k+1}} - P_k(x) - (x^{f_{k+1}} - x^{f_{k+1}+f_k} + x^{f_{k+1}+2f_k} - x^{f_{k+1}+3f_k})(1 + x^{f_k} - P_{k-1}(x))\\ &\quad = 1 + x^{f_{k+1}+4f_k} + x^{f_{k+1}}(1 - x^{f_k})(1 + x^{2f_k})P_{k-1}(x) - P_k(x)\\ &\quad = 1 + x^{f_{k+1}+4f_k} + (x^{f_{k+1}}(1 + x^{2f_k}) - 1)P_k(x) = 1 + x^{f_{k+1}+4f_k} + (x^{f_{k+1}} + x^{f_{k+2}} - 1)P_k(x).\end{aligned}$$

As above, using $F(x) = 1 - x\mathbf{a}(x) = \prod_{k=1}^{\infty}(1 - x^{f_k})$, we derive that

$$x(1 + x^{f_{k+1}+4f_k})\mathbf{a}(x) = (1 + x^{f_{k+1}+4f_k})\left(1 - \prod_{k=1}^{\infty}(1 - x^{f_k})\right).$$

Subtracting $1 + x^{f_{k+1}+4f_k}$ from both sides and then dividing by $P_k(x)$, we obtain that the inequality of the lemma is equivalent to the inequality

$$(1 + x^{f_{k+1}+4f_k}) \prod_{j=k+1}^{\infty} (1 - x^{f_j}) > 1 - x^{f_{k+1}} - x^{f_{k+2}}.$$

Plainly, $1 + x^{f_{k+1}+4f_k} > 1$, whereas the inequality $\prod_{j=k+1}^{\infty}(1 - x^{f_j}) > 1 - x^{f_{k+1}} - x^{f_{k+2}}$ follows from Lemma 6 (or from (15)) for any $k \in \mathbb{N}$. $\square$

5. Proof of Theorem 1

Fix $x \in (0, 1)$ and $\varepsilon > 0$. Let us consider the infinite word $\mathbf{s} = s_0s_1s_2s_3\ldots$ corresponding to the non-periodic sequence $s_0, s_1, s_2, s_3, \ldots$. In Section 3, we already proved that there is a proper suffix $\mathbf{s}_n = s_ns_{n+1}\ldots$, where $n \geqslant 1$, of $\mathbf{s}$ that satisfies $x\mathbf{s}_n(x) > \mathbf{a}(x) - \varepsilon = (1 - F(x))/x - \varepsilon$, unless $\mathbf{s}$ is the word of an alphabet $\{1, 2\}$. So suppose that $s_0, s_1, s_2, \ldots \in \{1, 2\}$. Setting $\mathbf{s}_{n,m} = s_ns_{n+1}\ldots s_{n+m-1}$, we obtain that

$$\mathbf{s}_n(x) + (-1)^{m+1}x^m\mathbf{s}_{n+m}(x) = s_n - xs_{n+1} + x^2s_{n+2} - \cdots + s_{n+m-1}(-x)^{m-1} = \mathbf{s}_{n,m}(x).$$

In particular, for each odd m, we have

$$\max\{\mathbf{s}_n(x), \mathbf{s}_{n+m}(x)\} \geqslant \frac{\mathbf{s}_{n,m}(x)}{1 + x^m} = \frac{\mathbf{s}_{n,m}(x)}{1 + x^{l(\mathbf{s}_{n,m})}}.$$

Since $\mathbf{a}(x) = (1 - F(x))/x$, for the proof of (8) it suffices to show that the word $\mathbf{s}$ contains a proper factor $\mathbf{u}$ of odd length such that

$$\mathbf{u}(x)/(1 + x^{l(\mathbf{u})}) > \mathbf{a}(x) - \varepsilon. \tag{16}$$

There are two possibilities: either the word $\mathbf{s}$ contains the proper factor 2111 or not. In the first case, it must contain at least one of the factors $\mathbf{a}_1\mathbf{a}_0^4 = 21111$ and $\mathbf{a}_1\mathbf{a}_0^3\mathbf{a}_1 = 21112$ as its proper factors. Since $\mathbf{a}_1\mathbf{a}_0^3\mathbf{a}_1(x) > \mathbf{a}_1\mathbf{a}_0^4(x)$, by Lemma 8, where $k = 1$, we obtain that $\mathbf{s}$ contains the proper factor $\mathbf{u}$ of odd length (which is $\mathbf{a}_1\mathbf{a}_0^3\mathbf{a}_1$ or $\mathbf{a}_1\mathbf{a}_0^4$) such that $\mathbf{u}(x)/(1 + x^{l(\mathbf{u})}) > \mathbf{a}(x)$ which is stronger than (16). Similarly, if $\mathbf{s}$ contains the proper factor $\mathbf{u} = \mathbf{a}_1\mathbf{a}_0\mathbf{a}_1 = 212$, we derive the same inequality, by Lemma 7, where $k = 1$. We can thus assume that the word $\mathbf{s}$ has a proper suffix composed from the blocs $\mathbf{a}_2 = 211$ and $\mathbf{a}_1 = 2$ only. By abuse of notation, we shall use the same notation $\mathbf{s}$ for this proper suffix of $\mathbf{s}$.

We can continue in the same manner as follows. Fix $k \in \mathbb{N}$. Assume $\mathbf{s}$ is composed from the blocks $\mathbf{a}_k$ and $\mathbf{a}_{k-1}$ only. Suppose that $\mathbf{s}$ contains the factor $\mathbf{u} = \mathbf{a}_k\mathbf{a}_{k-1}\mathbf{a}_k$ of odd length. It follows, by Lemma 7, that $\mathbf{u}(x)/(1 + x^{l(\mathbf{u})}) > \mathbf{a}(x)$. Since the word $\mathbf{a}_k$ begins with $\mathbf{a}_{k-1}$ for each $k \geqslant 2$, assuming that $\mathbf{s}$ contains the factor $\mathbf{a}_k\mathbf{a}_{k-1}^3$, we derive that it contains the factor $\mathbf{u} = \mathbf{a}_k\mathbf{a}_{k-1}^4$ of odd length. Now, by Lemma 8, we obtain again that $\mathbf{u}(x)/(1 + x^{l(\mathbf{u})}) > \mathbf{a}(x)$. So without loss of generality we can assume that $\mathbf{s}$ is composed from the blocks $\mathbf{a}_k$ and $\mathbf{a}_k\mathbf{a}_{k-1}^2 = \mathbf{a}_{k+1}$ only.

Since this is true for every $k \in \mathbb{N}$ and $\lim_{k\to\infty} \mathbf{a}_k = \mathbf{a}$, we can take k so large that

$$\mathbf{a}_k(x)/(1 + x^{f_{k+1}}) > \mathbf{a}(x) - \varepsilon.$$

But the word $\mathbf{s}$ contains the factor $\mathbf{a}_k$, where $l(\mathbf{a}_k) = f_{k+1}$, so this inequality gives (16) with $\mathbf{u} = \mathbf{a}_k$ of odd length. In fact, this is the only place, where we do need ε on the right-hand side of (16) and so on the right-hand side of (8) too. The proof of the theorem is completed. $\square$

6. Non-periodicity and transcendence

Let α be an algebraic number with minimal polynomial $a_d x^d + \cdots + a_0 \in \mathbb{Z}[x]$, and let ξ be a real number. Clearly, $a_d \xi\alpha^{n+d} + \cdots + a_1\xi\alpha^{n+1} + a_0\xi\alpha^n = \xi\alpha^n(a_d\alpha^d + \cdots + a_0) = 0$. So

$$s_n = a_d\{\xi\alpha^{n+d}\} + \cdots + a_0\{\xi\alpha^n\} = -a_d[\xi\alpha^{n+d}] - \cdots - a_0[\xi\alpha^n] \tag{17}$$

is an integer, whose modulus is bounded from above by $|a_d| + \cdots + |a_0|$. Let us consider the sequence $s_0, s_1, s_2, \ldots$.

Lemma 9. *Suppose that α is a real algebraic number satisfying $|\alpha| > 1$. Let $\xi \neq 0$ be a real number, and let $\xi \notin \mathbb{Q}(\alpha)$ if $|\alpha| = \pm\alpha$ is a Pisot number or a Salem number. Then the sequence $s_0, s_1, s_2, \ldots$ is non-periodic.*

Recall that an algebraic integer $\alpha > 1$ is called a *Salem number* if its degree d is at least 4 and if its conjugates except for α and α^{-1} are all lying on the unit circle $|z| = 1$. Similarly, an algebraic integer $\alpha > 1$ is called a *Pisot number* if its conjugates except for α are all lying in the open unit disc $|z| < 1$.

For $\alpha > 1$ and $\xi > 0$ the proof of the lemma was given in [20]. The proof for $\alpha < -1$ and $\xi < 0$ is exactly the same. It worth remarking that, generally speaking, this extra condition $\xi \notin \mathbb{Q}(\alpha)$ if $|\alpha|$ is a Pisot or a Salem number is necessary. See the papers [22,41], respectively.

For $\alpha = b = -p/q$, where $p > q \geqslant 1$ are coprime integers, the equality (17) becomes

$$s_n = q\{\xi b^{n+1}\} + p\{\xi b^n\} = q\{\xi(-p/q)^{n+1}\} + p\{\xi(-p/q)^n\} \in \{0, 1, 2, \ldots, p + q - 1\}. \tag{18}$$

Since $|b| = p/q$, according to the definitions of Pisot and Salem numbers, the integers $2, 3, 4, \ldots$ are Pisot numbers, whereas all other positive rational numbers greater than 1 are neither Pisot nor Salem numbers. So we need to add an extra condition $\xi \notin \mathbb{Q}$ in case $|b| \in \mathbb{N}$ in order to be sure that the sequence $s_0, s_1, s_2, \ldots$ were non-periodic. The next corollary is the only part of Lemma 9 which will be used below. (This corollary can be proved directly; see Lemma 2 in [24].)

Corollary 10. *Let $p > q \geqslant 1$ be two coprime integers. Suppose that $\xi \neq 0$ if $p > q > 1$ and that $\xi \notin \mathbb{Q}$ if $p > q = 1$. Then the sequence $s_0, s_1, s_2, \ldots$ defined in* (18) *is non-periodic.*

In the proof of Theorem 4 we shall use the following transcendence result.

Lemma 11. *If $g \geqslant 2$ is an integer then the number $F(1/g) = \prod_{k=1}^{\infty}(1 - g^{-(2^k + (-1)^{k-1})/3})$ is transcendental.*

Proof. By (3) and (5), we have that $F(1/g) = w_0 - w_1 g^{-1} + w_2 g^{-2} - w_3 g^{-3} + \cdots$. So

$$1 - F(1/g) = w_0 - F(1/g) = (gw_1 - w_2)g^{-2} + (gw_3 - w_4)g^{-4} + (gw_5 - w_6)g^{-6} + \cdots$$

is an expansion of the number $1 - F(1/g)$ in base g^2. The digits of this expansion $g_k = gw_{2k-1} - w_{2k}$, $k = 1, 2, \ldots$, belong to the set $\{g - 2, g - 1, 2g - 2, 2g - 1\}$. We know that the sequence $w_1, w_2, w_3, \ldots$ is non-periodic. Hence the sequence $g_1, g_2, g_3, \ldots$ is non-periodic too (see Lemma 3 in [20]). It follows that the number $1 - F(1/g)$ is irrational.

In order to prove that the number $1 - F(1/g)$ is transcendental, we shall use a recent result of Adamczewski and Bugeaud [2] (see also [1]), although this could also be derived from some earlier weaker results. Recall that, for each $n \in \mathbb{N}$, the *complexity* $p(\mathbf{u}, n)$ of the word $\mathbf{u}$ denotes the number of distinct factors of length n that occur in $\mathbf{u}$. (The notation p for the complexity has nothing to do with the integer p of Theorem 2 and is used in the proof of this lemma only.) According to [2], the number $\sum_{j=1}^{\infty} u_j u^{-j}$, where $u \geqslant 2$ is an integer, $u_j \in \{0, 1, \ldots, u - 1\}$, and $p(\mathbf{u}, n) \leqslant cn$ for each $n \in \mathbb{N}$, where $\mathbf{u} = u_1 u_2 u_3 \ldots$ and c is an absolute constant, is either transcendental or rational.

Let us estimate the complexity of the word $\mathbf{a} = w_1 w_2 w_3 \ldots$. As noticed by the referee, it is $\mathrm{O}(n)$, because $\mathbf{a}$ is the fixed point of a primitive morphism. On the other hand, it is easy to estimate the complexity of $\mathbf{a}$ explicitly. Clearly, $p(\mathbf{a}, 1) = 2$, $p(\mathbf{a}, 2) = 4$ and $p(\mathbf{a}, 3) = 7$, because $\mathbf{a}$ contains no factors 111 and 212. For any $n \geqslant 4$, we choose $k \geqslant 3$ for which $f_k \leqslant n < f_{k+1}$. All possible distinct factors of $\mathbf{a}$ clearly occur in the prefix $\mathbf{a}_{k+2} = \mathbf{a}_k \mathbf{a}_{k-1} \mathbf{a}_{k-1} \mathbf{a}_k \mathbf{a}_k$ of $\mathbf{a}$. Thus

$$p(\mathbf{a}, n) \leqslant p(\mathbf{a}, f_{k+1}) \leqslant l(\mathbf{a}_k \mathbf{a}_{k-1}^2 \mathbf{a}_k) = 2f_{k+1} + 2f_k = f_{k+2} + f_{k+1} = 2^{k+1} \leqslant 7f_k \leqslant 7n.$$

Note that this inequality holds for each $n \geqslant 1$.

In particular, given $g \geqslant 2$, by setting $\mathbf{g}=g_1g_2g_3\dots$, where $g_k=gw_{2k-1}-w_{2k}$, we obtain that $p(\mathbf{g}, n) \leqslant p(\mathbf{a}, 2n) \leqslant 14n$. Combining this inequality with the fact that $1-F(1/g) \notin \mathbb{Q}$, we obtain via [2] that $1-F(1/g)$ is transcendental. $\square$

Recall that, by a result of Cobham [17], the complexity $p(\mathbf{s}, n)$ of every automatic sequence $\mathbf{s}$ is bounded from above by cn.

7. Proofs of the results about fractional parts

Proof of Theorem 2. Suppose that $\xi \neq 0$ is a real number. Set $b=-p/q$, where $p>q$ are coprime positive integers and where q is allowed to be 1 in the case when $\xi \notin \mathbb{Q}$. Setting $x_n=[\xi b^n]$ and $y_n=\{\xi b^n\}$, we have $b(x_n+y_n)=(-p/q)(x_n+y_n)=x_{n+1}+y_{n+1}$. Thus $s_n=py_n+qy_{n+1} \in \{0,1,\dots,p+q-1\}$ as in (18). Furthermore, since $\{\xi b^n\}=0$ for at most finitely many $n \in \mathbb{N}$ (see, e.g., [24]), we obtain that $s_n>0$ for $n>n_0$. Hence $s_n \in \{1,2,\dots,p+q-1\}$ for each $n \geqslant n_0$. Next, the repeated application of $y_n=-qy_{n+1}/p+s_n/p=y_{n+1}/b+s_n/p$ yields

$$y_n=(s_n+s_{n+1}b^{-1}+\dots+s_{n+m-1}b^{-m+1})/p+y_{n+m}b^{-m}$$

for any $m \in \mathbb{N}$. Letting $m \to \infty$, we obtain that

$$y_n=\frac{1}{p}\sum_{j=0}^{\infty} s_{n+j}b^{-j}. \tag{19}$$

Put $r=q/p=-1/b$. Then (19) combined with Corollary 10 and Theorem 1 implies that, for any $\varepsilon>0$, the inequality

$$py_n=\sum_{j=0}^{\infty} s_{n+j}(-r)^j>(1-F(r))/r-\varepsilon$$

holds for infinitely many $n \in \mathbb{N}$. Thus $y_n=\{\xi b^n\}>(1-F(q/p))/q-\varepsilon/p$ for infinitely many $n \in \mathbb{N}$. Hence $\limsup_{n\to\infty}\{\xi b^n\} \geqslant (1-F(q/p))/q$. On replacing ξ by $-\xi$ and using $\{x\}=1-\{-x\}$ for $x \notin \mathbb{Z}$ we derive from this that the sequence $\{\xi b^n\}$, $n=0,1,2,\dots$, has a limit point smaller than or equal to $1-(1-F(q/p))/q$. The same trivially holds for $\xi=0$ too. This proves Theorem 2. $\square$

Proof of Theorem 4. We already proved the required bounds (7) in the proof of Theorem 2, so it remains to prove (i) and (ii). By Lemma 11, the number $\xi_b=-bF(-1/b)$ is transcendental. In order to show that the inequality $\{\xi_b b^n\}<1-F(-1/b)$ holds for every integer $n \geqslant 0$, using (1), (3) and (5), we first find that

$$\xi_b b^n=-bF(-1/b)b^n=-b^{n+1}\sum_{j=0}^{\infty} w_j b^{-j}=-b^{n+1}\sum_{j=0}^{n+1} w_j b^{-j}-b^{n+1}\sum_{j=n+2}^{\infty} w_j b^{-j}.$$

It follows that, for each integer $n \geqslant 0$, the fractional part

$$\{\xi_b b^n\}=-b^{n+1}\sum_{j=n+2}^{\infty} w_j b^{-j}=\sum_{j=0}^{\infty} w_{n+2+j}(-1)^j|b|^{-1-j}$$

is smaller than $\sum_{j=0}^{\infty} w_{1+j}(-1)^j|b|^{-1-j}=1-F(1/|b|)=1-F(-1/b)$. Indeed, since $\mathbf{a}=w_1w_2w_3\dots>\mathbf{s}=w_{n+2}w_{n+3}w_{n+4}\dots$ (recall that $\mathbf{a}$ is greater than its proper suffix with respect to the order $>$ introduced in Section 3), we obtain that $\mathbf{a}(1/|b|) \geqslant \mathbf{s}(1/|b|)$, because $1/|b| \leqslant 1/2$. Furthermore, $\mathbf{a}(1/|b|)>\mathbf{s}(1/|b|)$, because no suffix of $\mathbf{a}$ is of the form $(12)^\infty$. Thus

$$1-F(-1/b)=|b|^{-1}\mathbf{a}(1/|b|)>|b|^{-1}\mathbf{s}(1/|b|)=\sum_{j=0}^{\infty} w_{n+2+j}(-1)^j|b|^{-1-j}.$$

Now, using $\xi_b \notin \mathbb{Q}$ and $\{\xi_b b^n\} < 1 - F(-1/b)$ which was just proved above, we derive that

$$\{-\xi_b b^n\} = 1 - \{\xi_b b^n\} > 1 - (1 - F(-1/b)) = F(-1/b)$$

for each integer $n \geqslant 0$. This completes the proof of (ii). $\square$

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